

Gyrotropic Percolation

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Spiral (gyrotropic) percolation which is related to the behavior of an electron system in strong magnetic fields is studied. It is shown that the scaling behavior area near the percolation threshold is anomalously narrow. The percolation threshold is higher than in a system with usual isotropic percolation (i.e., at higher concentrations of undamaged structure elements). Our old value of the critical exponent of the correlation length is corrected.

The following is the translation from Russian of my old paper (1986) published in a poorly known (though it is translated into English [1]) Russian journal [1]. For this reason the paper seems to escape notice. (Note that Eq. (1) is written in a more compact form and misprints are corrected, a style of the translation is also corrected.)

Before the beginning of the main text, I want to mention several very important subsequent papers on the spiral percolation [2] (P. Ray and I. Bose) and [3, 4, 5] (S. B. Santra and I. Bose). These authors have come to idea of this type of percolation from their studies of spiral lattice animals [6, 7, 8, 9]. One can find modern results – the best values of critical exponents and the thresholds for different lattices obtained by direct calculations – and fresh references concerning the problem under consideration in the mentioned papers and in interesting lectures of I. Bose [10]. I am grateful to Prof. I. Bose who have informed me about these valuable papers. Note that all other references are only to papers published before [1].

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The probability of electron jumps in one or another direction with respect to the direction of the preceding jump is anisotropic function of the corresponding angle [11, 12] for systems with hopping conduction in an applied magnetic field. This anisotropy is enhanced in classically strong fields and can lead to changes in the percolation properties of the material. We shall consider the simplest lattice model to study such systems.

Let us consider, for example, the bond problem for a two-dimensional lattice. A particle is allowed to move from site to site via bonds and a fraction of such bonds is broken randomly. We shall assume that each subsequent displacement of the particle via allowed bonds relative to its preceding displacement can take place only in two ways: forward or to the left, i.e., right-hand turns are forbidden.

If the direction of each subsequent step were determined by a random-number generator, we would have a problem of a spiral random walk (see, for example, [13]). As for an ordinary random walk [14, 15], we can study a random walk of this type also in a directed system. However, we shall consider a different problem, i.e., what values of the concentration of the unbroken bonds are required for a particle to pass from one edge of a lattice to the other one if we forbid the right-hand turns. We are also interested in the critical properties of such a process which is called gyrotropic (or spiral) percolation. (Of course, systems with higher dimensionality can also be considered.)

The crudest naive estimate of the percolation threshold for gyrotropic percolation on a hypercubic lattice of dimensionality D yields $u_c \sim u_{c0}[1 - 1/(2D - 1)]^{-1}$, where u is the concentration of the unbroken bonds and u_{c0} is the percolation threshold for ordinary isotropic percolation. Enumerating random configurations on a square lattice (the bond problem), we obtain $u_c = 0.69 \pm 0.03$ ($u_{c0} = 0.5$).

We shall use the real space renormalization group approach in our further consideration (position space renormalization group). The percolation problem for a system of conductors and randomly oriented diodes (randomly directed percolation) was first studied by S. Redner [16, 17, 18, 19]. Such a problem is a generalization of the directional percolation problem [20, 21, 22]. It will be shown here, that in the case under consideration, the renormalization group transformations induce randomly directed bonds like bonds ab initio presenting on a lattice for the randomly directed percolation problem.

The renormalization group transformations transform clusters consisting

of several bonds to new effective bonds, so in fact we apply sequentially a coarsening procedure. It follows from Fig. 1 that even a small cluster on a square lattice contains configurations of isotropic bonds which induce upon renormalization transformation oriented bonds. Let us denote by q the probability of appearance of such bonds. The probability of finding a broken bond is then given by $1 - u - 2q$, since there are possible two bond orientations. Of course, the bare value of the above probability should be $q(0) = 0$!

To make the problem completely local, we assume that *a particle can not come to a stop and begin to move backwards*. Note, it is a principal assumption of the problem under consideration!

To obtain the required recursive renormalization group relations for the probabilities defined above, we have to enumerate $(1+1+2)^N$ possible configurations, where N is the number of bonds in the cluster used. Large clusters containing at least several tens of bonds are required for quantitative analysis. Thus, the enumeration is possible only by the Monte Carlo method and a large computer is required.

Since we are interested only in qualitative features of this type of percolation, it is sufficiently to consider a small cluster consisting of six bonds on a hierarchical lattice (see Fig. 2). Even using this minimal suitable cluster (available for manual enumeration of configurations), one can find that the renormalization transformations induce oriented bonds. Although the cluster considered does not give the exact isotropic percolation threshold (like a cluster consisting of four bonds with a scaling transformation coefficient $1/\sqrt{2}$), but enables us to obtain after the recursive transformation both oriented and nonoriented bonds.

Such a cluster admits a large enough number of percolation channels. This requirement is not satisfied by the simplest self-dual cluster consisting of five bonds [21]. The main features of the corresponding recursive relations and the qualitative positions of their fixed points remain unchanged for larger clusters.

Let us write the recursive relations immediately (in [1] these relations were presented in an expanded form):

$$\begin{aligned} u' &= u^2[1 - (u + q)^2]^2 + (u + q)^4[3 - 2(u + q)^2] , \\ q' &= [1 - (u + q)^2]^2[2(u + q)^2 - u^2] . \end{aligned} \tag{1}$$

The corresponding phase diagram is shown in Fig. 3. The gyrotropic percolation threshold is $u_c = (\sqrt{5} - 1)/2 = 0.618$. Using the same cluster for isotropic percolation, one obtains the following recursive relation:

$$u' = 1 - (1 - u^2)^3, \quad (2)$$

and $u_{c0} = (3 - \sqrt{5})/2 = \sqrt{u_c} = 0.389$. Such a shift of the percolation threshold leads to lowering of a conductivity of the structure relative to the usual one. This reduction corresponds to increasing of the magnetoresistance with increasing magnetic field [23].

The recursive relations (1) have only three fixed points: $(0, 0)$, $(1, 0)$, and $(u^*, q^*) = ((3 - \sqrt{5})/2, \sqrt{5} - 2) = (0.382, 0.236)$. The equation for one of the separatrices is $u + q = (\sqrt{5} - 1)/2 = 0.618$. The coordinates of the fixed point (u^*, q^*) are also related by $2u^* + q^* = 1$. The maximum of the second separatrix $q = \sqrt{u} - u$ is at the point $(u_{max}, q_{max}) = (1/2, 1/4)$.

Beautiful phase diagrams very similar to that shown in Fig. 3 but much more symmetrical ones were first obtained in papers of S. Redner and coauthors where randomly directed percolation was studied. However, in the Redner's problem, oriented bonds are not induced by the renormalization if bare $q(0)$ is zero. In contrast to problem [16, 17, 18, 19] the threshold point under consideration $(u_c, 0)$ is not a fixed point. In fact, the critical behavior is determined by the fixed point (u^*, q^*) . It is easy to see that one gets after n transformations (1) the following deviation

$$u(n) - u^* = C(n)(u(n=0) - u_c). \quad (3)$$

Here, $C(n)$ depends only on n , $u(n=0) - u_c \ll 1$, and n is chosen such one that $u(n) - u^* \ll 1$. Thus, the scaling exponent ν for the correlation length $\xi \sim |u - u_c|^{-\nu}$ can be found by linearization of Eq. (2) near the fixed point (u^*, q^*) :

$$\begin{aligned} \begin{pmatrix} u' - u^* \\ q' - q^* \end{pmatrix} &= \begin{pmatrix} 13 - 5\sqrt{5} & 2(3 - \sqrt{5}) \\ -(7 - 3\sqrt{5}) & 0 \end{pmatrix} \begin{pmatrix} u' - u \\ q' - q \end{pmatrix} \\ &= \begin{pmatrix} 1.820 & 1.528 \\ -0.292 & 0 \end{pmatrix} \begin{pmatrix} u' - u \\ q' - q \end{pmatrix}. \end{aligned} \quad (4)$$

and $\nu \propto 1/\log(\lambda_1)$. Here $\lambda_1 = 2(3 - \sqrt{5}) = 1.528$ and $\lambda_2 = 7 - 3\sqrt{5} = 0.292$ are exigent values of Eq. (4). The value $\lambda = \partial u'/\partial u(u_{c0}) = 1.681$ may

be obtained for isotropic percolation using the *same* cluster (see Eq. (2)). Thus The critical exponent ν for gyrotropic percolation is 1.225 of ν value for ordinary percolation.

For the unrenormalized values of $u(0)$ satisfying $u(0) - u_c > 10^{-2}$, the phase trajectory passes far from the fixed point (u^*, q^*) and Eq. (4) will never be valid. Thus, the scaling region is much narrower than for isotropic percolation. Such an anomalously narrow region hardly can be observed for realistic systems. A similar narrowing of the scaling region should occur for systems in which the renormalization transformation induce bonds of new type.

We have studied only a strongly gyrotropic situation. Under more realistic conditions when right-hand turns are possible although less probable than left-hand turns, the increase of the percolation threshold is not so large but should be noticeable.

Finally, we would like to point out that larger clusters cutted from realistic lattice systems could lead to different shapes of separatrices as in the case of randomly directed percolation [16, 17, 18, 19].

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FIGURE CAPTIONS

Fig. 1. Example of a configuration which induces oriented renormalized bonds in the problem of gyrotropic percolation. Right-hand turns are forbidden. The broken lines denote broken bonds in a cluster. The arrow indicates possible orientation of percolation.

Fig. 2. Clusters used at the derivation of recursive relations (1) and (2).

Fig. 3. Phase diagram of the recursive relations (1). Compare with the corresponding phase diagrams of S. Redner et al for randomly directed percolation [16, 17, 18, 19].

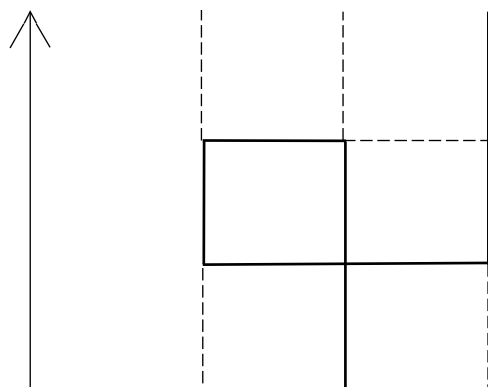


Fig. 1

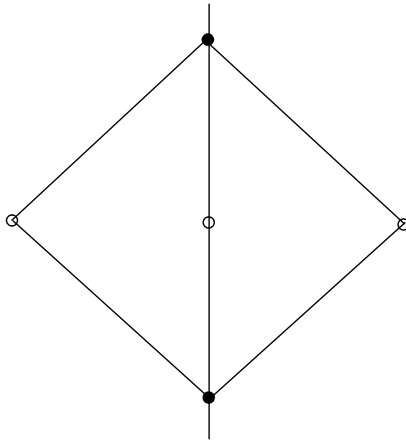


Fig. 2

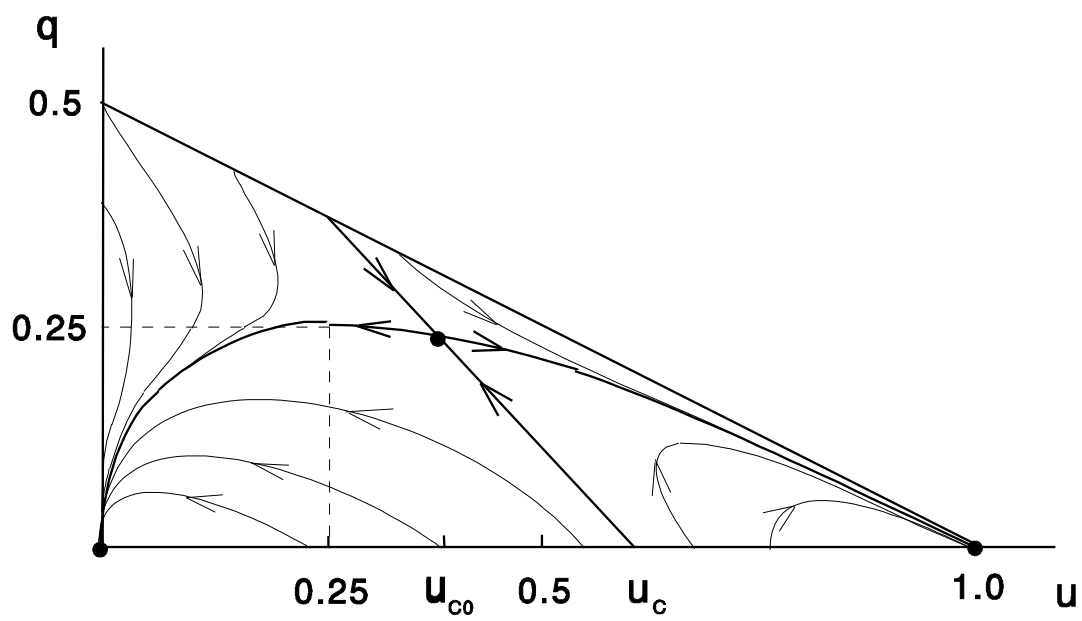


Fig. 3